SOLUTIONS OF EXERCISES.

27

THE NUMBER of points common to three surfaces of the mth, nth, and pth degrees being in general mnp, find the co-ordinates of all the real and imaginary points of intersection of the three surfaces, $y^2 + x^2 + z^2 = 4x$, $x^2 = yz$, and $y^2 = x^4$.
[*H. A. Newton.*]

SOLUTION.

The three surfaces $x^2 + y^2 + z^2 - 4x$, $x^2 - yz$, $x^4 - y^2$ are respectively a sphere, a hyperbolic parabaloid, and a pair of parabolic cylinders $x^2 \pm y$. The parabolic cylinders intersect the hyperbolic paraboloid in the planes $z \pm 1$ and at infinity. These planes cut the sphere in the circles $x^2 + y^2 - 4x + 1$ which intersect the parabolas $x^2 \pm y$ in the points whose abscissæ are the roots of

$$x^4 + x^2 - 4x + 1 = 0;$$

that is $x_1 = 0.27$; $x_2 = 1.25$; $x_3 = -0.76 + 1.54i$; $x_4 = -0.76 - 1.54i$.

The corresponding values of y are

$$y_1 = \pm 0.07;\ y_2 = \pm 1.56;\ y_3 = \mp (1.79 + 2.34i);\ y_4 = \mp (1.79 - 2.34i).$$

In addition to these eight intersection points, four real and four imaginary, there are altogether at infinity eight other intersection points, which make up the full number of sixteen $(2 \times 2 \times 4 = 16)$. [*Wm. M. Thornton.*]

46

FIND an equation that shall express approximately the surface of an egg shell. By means of the variations of not more than three constants in the equations, express the various sizes and shapes of eggs. [*H. A. Newton.*]

SOLUTION.

Assume a fixed circle of radius r and centre C, and a fixed point O. Join O to any point P on the circumferencc and on PO lay off $PM = c$. If then

$$OM = \rho,\quad COM = \theta,\quad OC = a,$$

it easy to show by projecting OC on OP that

$$\rho = a \cos\theta - c - \sqrt{(r^2 - a^2 \sin^2\theta)}.$$

This equation represents the locus of M referred to O as pole. This locus is an

ovoid curve whose rotation about OC generates the egg shell. By varying the constants a,c,r we obtain the various sizes and shapes of eggs. In particular, if $c = 0$ the curve is the base circle and the egg is spherical.

[*J. E. Hendricks.*]

47

Two bulls arriving on opposite banks of a river a rods wide, plunge into the water at the same instant and swim toward each other until they meet. One bull can swim m miles and the other n $(m > n)$ miles an hour in still water, and the velocity of the river is v miles an hour. Required the equation to the curve described and the distance swum by each bull and the location of the point of meeting. [*Artemas Martin.*]

SOLUTION.

Let A,B be the bulls; P,Q their positions at any instant; and P_0,Q_0 their initial positions. Then since the velocity of the stream is everywhere constant, PQ will remain always parallel to P_0Q_0; and the bulls will meet after t seconds where

$$mt + nt = a \operatorname{cosec} \beta,$$

a being the breadth of the stream and β the angle between P_0Q_0 and the bank.

Since the component velocities of A,B are constant the paths will be straight. The resultant velocities will be

$$u = \surd(m^2 + v^2 - 2mv \cos \beta),$$
$$w = \surd(n^2 + v^2 + 2nv \cos \beta),$$

and the spaces described will be ut, wt.

[*J. E. Hendricks.*]

48

In the angle A a point P is given. Construct a triangle ABC whose base BC shall contain P while the sum of the sides AB and AC equals a given length. [*W. M. Thornton.*]

SOLUTION.

It is readily shown that the line

$$\frac{x}{a} + \frac{y}{k - a} = 1$$

is tangent to the parabola

$$x^2 - 2xy + y^2 - 2k(x + y) + k^2 = 0.$$

Setting $AB + AC = K$, a knowledge of the fundamental properties of tangents to a parabola makes the following construction self-explanatory:—

Lay out on AB, AC, respectively, AX, AY, each equal to K. Draw AN through N, the middle point of XY. V, the middle point of AN, is the principal vertex of the parabola mentioned above. Draw VZ perpendicular to AV, cutting AX in Z; then ZS perpendicular to AX, cutting AN in S, the focus of the parabola. Over SP as a diameter draw a circle cutting VZ in Q,R. Draw PQ, PR; these are tangents to the parabola and cut AX, AY in B,C, giving two constructions.

[*R. D. Bohannan.*]

EXERCISES.

51

A QUADRILATERAL, both circumscriptible and inscriptible, has a given perimeter $4s$ and one given angle α. Show that its area is

$$2s^2 : (\operatorname{cosec} \alpha + \operatorname{cosec} x)$$

where x is one of the angles adjacent to α. And find x when the area is greatest.

[*W. M. Thornton.*]

52

FIND the height to which the Washington Monument must be built so that a body placed on top of it would have no weight. [*A. Hall.*]

53

SHOW that when $x = \frac{2\pi}{13}$

$$(\cos x + \cos 5x)(\cos 2x + \cos 3x)(\cos 4x + \cos 6x) = -\tfrac{1}{8}.$$

[*R. H. Graves.*]

54

SHOW that if in a plane triangle ABC

$$\cos A + \cos B + \cos C = \sqrt{2},$$

the centre of the circumscribed circle lies on the circumference of the inscribed circle.

[*R. D. Bohannan.*]